MAT185 Linear Algebra
Assignment 2

Instructions:

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3. **Show your work and justify your steps** on every question but do not include extraneous information. Put your final answer in the box provided, if necessary. We recommend you write draft solutions on separate pages and afterwards write your polished solutions here on this template.
4. **You must fill out and sign the academic integrity statement below**; otherwise, you will receive zero for this assignment.

Academic Integrity Statement:

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I confirm that:

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- In particular, I have read and understand the rules for collaboration, and permitted resources on assignments as described in subsection II of the the aforementioned document. I have not violated these rules while completing and writing this assignment.
- I understand the consequences of violating the University's academic integrity policies as outlined in the Code of Behaviour on Academic Matters. I have not violated them while completing and writing this assignment.

By signing this document, I agree that the statements above are true.

Signatures: 1) ________

2) Sharn

1. Let W be the subspace of ${}^n\mathbb{R}^n$ defined by

$$W = \{A = [a_{ij}] \in {}^n\mathbb{R}^n \mid \sum_{j=1}^{n} a_{ij} = 0, \text{ for every } i, \text{ and } \sum_{i=1}^{n} a_{ij} = 0, \text{ for every } j\}$$

What is $\dim W$? Suppose $\underline{A} \in \underline{W}$

For $\sum_{j=1}^{n} a_{ij} = 0 \ \forall i$, $a_{1j}, a_{2j}, \dots a_{n-1j}, a_{nj}$ must be linearly dependent.

$\lambda_1 a_{1j} + \lambda_2 a_{2j} + \dots + \lambda_{n-1} a_{n-1j} + \lambda_n a_{nj} = 0$, but since

① $a_{1j} + a_{2j} + \dots + a_{n-1j} + a_{nj} = 0$, $\lambda_1, \lambda_2, \dots, \lambda_{n-1}, \lambda_n \in \mathbb{R}$

Because $\underline{A} \in \underline{W} \Rightarrow \tilde{\underline{A}} \in \underline{W} \Rightarrow$ therefore for $\sum_{i=1}^{n} a_{ij} = 0 \ \forall j$,

$a_{i1}, a_{i2}, \dots, a_{in-1}, a_{in}$ must be linearly dependent.

Rearranging equation ①, we can determine a general form for the nth term of each row & column:

$$-a_{nj} = a_{1j} + a_{2j} + \dots + a_{n-1j} = \sum_{i=1}^{n-1} a_{ij} \Rightarrow a_{nj} = -\sum_{i=1}^{n-1} a_{ij}$$

and

$$-a_{in} = a_{i1} + a_{i2} + \dots + a_{in-1} = \sum_{j=1}^{n-1} a_{ij} \Rightarrow a_{in} = -\sum_{j=1}^{n-1} a_{ij}$$

Thus, the general form of $\underline{A} \in \underline{W}$ is:

$$\underline{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n-1} & -\sum_{i=1}^{n-1} a_{ij} \\ a_{21} & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ -\sum_{i=1}^{n-1} a_{ij} & \cdots & & \cdots & \sum_{i,j=1}^{n-1} a_{ij} \end{bmatrix}$$

Thus, we can see that the nth row & column are always linear combinations of the previous terms in the row or column.

So, the dimension of $\underline{W}$, $\dim \underline{W}$, which is the number of vectors in any of its bases is $(n-1)^2$.

Since, a basis is a set of linearly independent vectors which span the vector space. And the number of linearly independent vectors in $\underline{W}$ will be $(n-1)(n-1) = (n-1)^2$, as the last row & column of $\underline{A}$ will always be a linear combination of the previous terms in the row or column.

$\therefore \dim \underline{W} = (n-1)^2$

2. Let X be any non-empty set and define $F(X) = \{f \mid f : X \to \mathbb{R}\}$. Define vector addition and scalar multiplication in $F(X)$ in the usual way:

$$(f+g)(x) = f(x) + g(x)$$
$$(cf)(x) = cf(x), \quad c \in \mathbb{R}$$

for all $x \in X$. Then $F(X)$ is a vector space (cf. Medici, Section 4.2, page 105).

Let n be a positive integer. If $X = \{1, 2, 3, \ldots, n\}$, what is $\dim F(X)$?

According to theorem V. (Medici, pg. 143), Existence of bases: let $\underline{F(X)}$ be a vector space spanned by a finite set of vectors. Then every linearly independent set of vectors in $\underline{F(X)}$ can be extended to a basis for $\underline{F(X)}$.

One possible basis for $\underline{F(X)}$ is $f_i(x) = \begin{cases} 1 & \text{if } x = i \\ 0 & \text{if } x \neq i \end{cases}$

$\Rightarrow$ that if $i = 1$, $f_1(x) = \{1, 0, 0, \ldots, 0\}$

if $i = 2$, $f_2(x) = \{0, 1, 0, \ldots 0\}$

if $i = k$, $f_n(x) = \{0, 0, \ldots 1\}$

Therefore, this is a basis as the set of vectors span $\underline{F(x)}$ and are linearly independent.

Thus, they can be expressed as:

$a\{1, 0, 0, \ldots, 0\} + b\{0, 1, 0, \ldots, 0\} + \ldots + k\{0, 0, \ldots, 1\} = 0$, ①

where $a, b, \ldots, n = 0$.

Thus, it is clear that the number of vectors in the basis is k.

$\Rightarrow k = n$ since there are n-values in the range of $f_i(x)$

e.g. $\{f_i(1), f_i(2), f_i(3), \ldots f_i(n)\}$ the basis indicates that there is one vector for each x in $f_i(x)$.

So, therefore to span $\underline{F(X)}$, there must be n vectors in the basis.

$\therefore$ Therefore, the dimension of $\underline{F(X)}$, $\dim \underline{F(X)} = n$

3. Let U and W be subspaces of a vector space V. If $\dim U = k$, and $\dim W = l$, prove that $\dim(U+W) \le k+l$. Under what condition would $\dim(U+W) = k+l$?

PART A

We want to prove $\dim(U+W) \le k+l$ where $\dim U = k$ and $\dim W = l$. If $U, W \subset V$ are subspaces of a vector space V.

Let $(v_1, \ldots, v_n)$ be a basis of $U \cap W$, where $v_1, \ldots, v_n$ are vectors shared by the subspaces U and W, which are linearly independent.

Then $\dim(U \cap W) = n$

Using Theorem V (medici pg 143)/Existence of Basis/Extend-Reduce Theorem redux: Every linearly independent set of vectors in a vector space can be extended to form a basis for the vector space. By this basis extension theorem, there exists $(u_1, \ldots, u_y) \in U$ and $(w_1, \ldots, w_z) \in W$ such that:

$(v_1, \ldots, v_n, u_1, \ldots, u_y)$ is a basis of U

and $(v_1, \ldots, v_n, w_1, \ldots, w_z)$ is a basis of W.

Therefore $\dim U = n + y = k$ and $\dim W = n + z = l$.

Using the basis of U and W found above; the basis of $U+W$ can be given by: $(v_1, \ldots, v_n, u_1, \ldots u_y, w_1, \ldots, w_z)$ [definition of a basis]

$$
\begin{aligned}
\text{Therefore, } \dim(U+W) &= n + y + z \\
&= n + y + z + (n - n) \quad \text{[Property of real numbers]} \\
&= (n+y) + (n+z) - n \quad \text{[Associativity]} \\
&= \dim U + \dim W - \dim(U \cap W) \\
&= k + l - \dim(U \cap W)
\end{aligned}
$$

This equation can be rearranged into:

$$\dim(U+W) + \dim(U \cap W) = k + l. \quad \text{Formula 1}$$

We know $\dim(U \cap W) \ge 0$, because U and W contain at least the zero vector that is shared in common. Because U and W are in the same vector space V and are subspaces, the $\dim(U+W) = 0$ if they only share the zero vector and $\dim(U \cap W) > 0$ if they share one or more non-zero linearly independent vectors.

If $\dim(U \cap W) \ge 0$ then for the equality to hold (in Formula 1)

$$\dim(U+W) \le k+l$$

Which proves the statement aforementioned.

PART B For $\dim(U+W) = k+l$, $\dim(U \cap W) = 0$ which means the only vector U and W can share is the zero vector.

$\dim(0) = 0$ [Because U and W are both subspaces of V]

4. Show that if

$$A = \sum_{i=1}^{k} \mathbf{x}_i \mathbf{y}_i^T$$

for some $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k \in {}^m\mathbb{R}$, and $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k \in {}^n\mathbb{R}$, then rank $A \leq k$.

We want to verify that for $A = \sum_{i=1}^{K} x_i y_i^T$, rank $A \leq K$.

For $x_1, x_2, \ldots, x_K \in {}^m\mathbb{R}$ and $y_1, y_2, \ldots, y_K \in {}^n\mathbb{R}$

Let $A_i = x_i y_i^T$, then $A = \sum_{i=1}^{K} x_i y_i^T = \sum_{i=1}^{K} A_i$.

We know that $x_i = \underset{m \times 1}{\begin{bmatrix} v_{i1} \\ v_{i2} \\ \vdots \\ v_{im} \end{bmatrix}}$ and $y_i = \underset{n \times 1}{\begin{bmatrix} z_{i1} \\ z_{i2} \\ \vdots \\ z_{in} \end{bmatrix}}$. Then $y_i^T = \begin{bmatrix} z_{i1} & z_{i2} & \cdots & z_{in} \end{bmatrix}$

for $v_{i1}, v_{i2}, \ldots v_{im} \in x_i$ and $z_{i1}, z_{i2}, \ldots, z_{in} \in y_i$.

Then $A_i = x_i y_i^T = \begin{bmatrix} v_{i1} z_{i1} & v_{i1} z_{i2} & \cdots & v_{i1} z_{in} \\ v_{i2} z_{i1} & v_{i2} z_{i2} & \cdots & v_{i2} z_{in} \\ \vdots & \vdots & & \vdots \\ v_{im} z_{i1} & v_{im} z_{i2} & \cdots & v_{im} z_{in} \end{bmatrix}$ where A_i is an $m \times n$ matrix.

Therefore from the matrix above we know that

$$\text{Col } A_i = \text{Span}\{x_i\}$$

Since $A = \sum_{i=1}^{K} A_i$ (from above), $\text{Col } A = \sum_{i=1}^{K} \text{Col } A_i = \text{Col } A_1 + \text{Col } A_2 + \ldots + \text{Col } A_K$

$$= \text{Span}\{x_1, x_2, \ldots x_K\}$$

Consider the two possible situations:

(a) If the set $\{x_1, x_2, \ldots x_K\}$ is linearly independent then it forms the minimal spanning set/basis of Col A. Therefore dim Col $A = K$.

dim Col A = rank $A = K$.

(b) If the set $\{x_1, x_2, \ldots, x_K\}$ is not linearly independent then dim Col $A < K$. Correspondingly rank $A < K$.

According to the fundamental theorem of calculus (FToLA), we know that in a vector space, the number of linearly independent vectors cannot exceed the number vectors spanning the vector space which means rank $A \not> K$.

Therefore combining the statements above we can conclude that:

$$\text{rank } A \leq K$$